

On the existence of finite critical trajectories in a family of quadratic differentials

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Abstract

In this note, we discuss the possible existence of finite critical trajectories connecting two zeros in a family of quadratic differentials satisfying some properties. We treat the cases of holomorphic and meromorphic quadratic differentials. In addition, we reprove some results about the support of limiting root-counting measures of the generalized Laguerre and Jacobi polynomials with varying parameters.

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1 Introduction and main results

Quadratic differentials appear in many areas of mathematics and mathematical physics such as orthogonal polynomials, moduli spaces of algebraic curves, univalent functions, asymptotic theory of linear ordinary differential equations etc...

One of the most common problems in the study of a given quadratic differential, is the existence or not of its short trajectories. In this note, we answer this question, under suitable assumptions.

In section 4, we present new proofs of the existence of short trajectories of quadratic differentials related to generalized Laguerre and Jacobi polynomials with varying parameters.

Let Ω be a non empty connected subset of \mathbb{C} , and $Q(z) = \prod_{k=1}^3 (z - a_k)^{m_k}$ be a polynomial that is with simple or double zeros ($m_k \in \{1, 2\}$). Let $a, b : \Omega \longrightarrow \mathbb{C} \setminus \{a_1, a_2, a_3\}$ be two continuous functions such that

$$\forall t \in \Omega, a(t) \neq b(t). \quad (1)$$

We consider the family of rational and polynomial functions R_t and P_t

$$\begin{aligned} R_t(z) &= \frac{(z - a(t))(z - b(t))}{Q(z)}, \\ P_t(z) &= (z - a(t))(z - b(t))Q(z). \end{aligned}$$

We denote $\mathcal{J}_{a(t), b(t)}$ the set of all Jordan arcs in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$ joining $a(t)$ and $b(t)$, and we suppose that there exists a continuous function (in the Hausdorff metric)

$$\Phi : \Omega \longrightarrow \mathcal{J}_{a(t), b(t)} \quad t \longmapsto \phi_t,$$

such that

$$\phi_t(0) = a(t), \phi_t(1) = b(t). \quad (2)$$

We assume that for some choice of branches of the square roots $\sqrt{R_t(z)}$ and $\sqrt{P_t(z)}$, ϕ_t satisfies

$$\Re \int_{\phi_t} \sqrt{R_t(z)} dz = 0; \quad (3)$$

$$\Re \int_{\phi_t} \sqrt{P_t(z)} dz = 0. \quad (4)$$

We consider the quadratic differentials

$$\begin{aligned} \varpi(R_t, z) &= -R_t(z) dz^2, \\ \varpi(P_t, z) &= -P_t(z) dz^2. \end{aligned}$$

Then, the following results hold

Proposition 1 *Under assumptions (1), (2), and (3), either, for any $t \in \Omega$, there exists exactly one short trajectory of the quadratic differential $\varpi(R_t, z)$ that connects $a(t)$ and $b(t)$, homotopic to ϕ_t in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$, or there does not exist any such trajectory for any $t \in \Omega$.*

Proposition 2 *With assumptions (1), (2), and (4), the set of all $t \in \Omega$ such that $\varpi(P_t, z)$ has a short trajectory connecting $a(t)$ and $b(t)$ is a closed subset of Ω .*

2 Basics of quadratic differentials

We first present some basics for quadratic differentials.

Definition 3 A rational quadratic differential on the Riemann sphere $\overline{\mathbb{C}}$ is a form $\varpi = \varphi(z)dz^2$, where φ is a rational function of a local coordinate z . If $z = z(\zeta)$ is a conformal change of variables then

$$\tilde{\varphi}(\zeta)d\zeta^2 = \varphi(z(\zeta))(dz/d\zeta)^2 d\zeta^2$$

represents ϖ in the local parameter ζ .

The *critical points* of ϖ are its zeros and poles; a critical point is *finite* if it is a zero or a simple pole, otherwise, it is *infinite*. All other points of $\overline{\mathbb{C}}$ are called *regular* points.

The horizontal trajectories (or just trajectories) are the zero loci of the equation

$$\Im \int^z \sqrt{\varphi(t)} dt = \text{const}, \quad (5)$$

or equivalently

$$\varphi(z) dz^2 > 0;$$

the vertical trajectories are obtained by replacing \Im by \Re in the equation above. The horizontal and vertical trajectories of ϖ produce two pairwise orthogonal foliations of the Riemann sphere $\overline{\mathbb{C}}$. A critical trajectory is a trajectory passing through a critical point. A finite critical trajectory or *short trajectory* is a critical trajectory connecting two finite critical points of ϖ , it will be called *unbroken* if it is not passing through other finite critical points except its two endpoints, otherwise, we call it *broken*. The set of finite and infinite critical trajectories of ϖ together with their limit points (critical points of ϖ) is called the *critical graph* of ϖ .

Notice that, if $z(t), t \in \mathbb{R}$ is a trajectory of (5), then the function

$$t \mapsto \Re \int^t \sqrt{\varphi(z(u))} z'(u) du$$

is monotone. For more details, we refer the reader to [13].

The local structure of the trajectories is as follow :

- At any regular point horizontal (resp. vertical) trajectories look locally as simple analytic arcs passing through this point, and through every regular point of ϖ passes a uniquely determined horizontal (resp. vertical) trajectory of ϖ ; these horizontal and vertical trajectories are locally orthogonal at this point.

- From every zero with multiplicity r of ϖ , there emanate $(r + 2)$ horizontal (resp. vertical) trajectories, and the angle between any two adjacent trajectories equals $\pi / (r + 2)$.
- At a simple pole there emanates only one trajectory (see Figure 1).
- At a double pole, the local behavior of the trajectories depends on the vanishing of the real or imaginary part of the residue; they have either the radial, the circular or the log-spiral form (Figure 2).
- At a pole of order r greater than 2, there are $(r - 2)$ asymptotic directions (called *critical directions*) spacing with equal angle $\frac{2\pi}{r-2}$, and a neighborhood \mathcal{U} , such that each trajectory entering \mathcal{U} and tends to this pole in one of the critical directions.

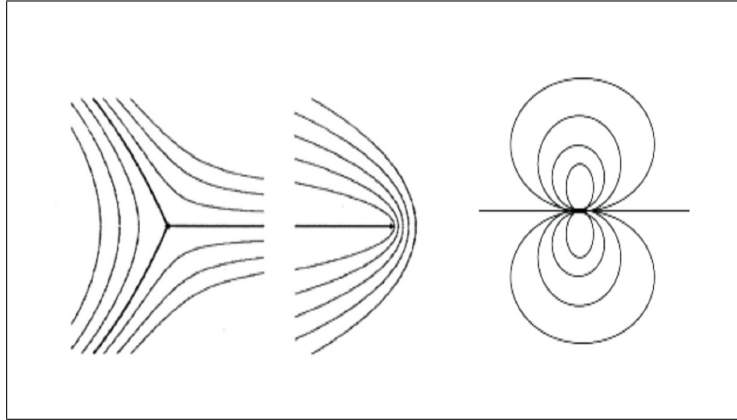


Figure 1: Structure of trajectories in a neighborhood of a simple zero (left), simple pole (middle), and a 4th order pole (right)..

The main trouble in the global behaviour of trajectories which are dense in some domains in \mathbb{C} comes from the so-called recurrent trajectory; Jenkins' three pole Theorem asserts that such a situation cannot happen for a quadratic differential that has at most three poles.

A necessary condition for the existence of a short trajectory connecting two finite critical points a and b of a quadratic differential $\varphi(z) dz^2$ is the existence of Jordan arc γ connecting a and b in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$, such that

$$\Im \int_{\gamma} \sqrt{\varphi(t)} dt = 0,$$

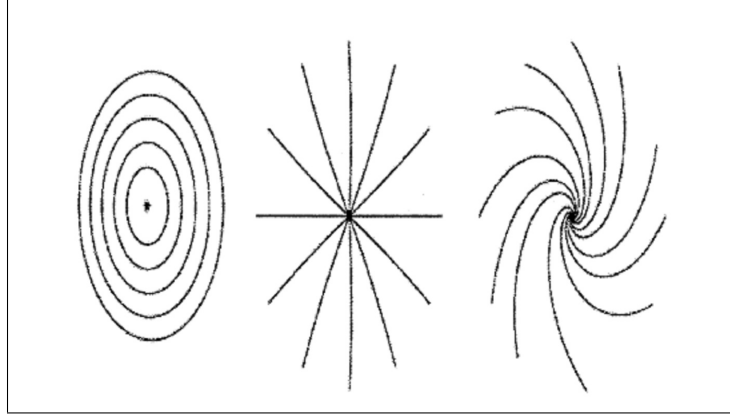


Figure 2: Structure of trajectories in a neighborhood of a double pole, circle form (left), radial form (middle), and log-spiral form (right).

but this condition is not sufficient. Indeed, Figure 3 illustrates the critical graph of the quadratic differential $Q(z) = -(z^4 - 1) dz^2$; in particular, there is no short trajectory connecting the zeros $\pm i$. However, if γ is an oriented Jordan arc joining $\pm i$ in $\mathbb{C} \setminus [-1, 1]$, and $\sqrt{z^4 - 1}$ is chosen in $\mathbb{C} \setminus ([-1, 1] \cup \gamma)$ with condition $\sqrt{z^4 - 1} \sim z^2, z \rightarrow \infty$, then from the Laurent expansion at ∞ of $\sqrt{z^4 - 1}$:

$$\sqrt{z^4 - 1} = z^2 + \mathcal{O}(z^{-2}), z \rightarrow \infty,$$

we deduce the residue of $\sqrt{z^4 - 1}$ at ∞ :

$$\text{res}_\infty(\sqrt{z^4 - 1}) = 0.$$

For $t \in [-1, 1] \cup \gamma$, we denote by $(\sqrt{t^4 - 1})_+$ and $(\sqrt{t^4 - 1})_-$ the limits from the $+$ -side and $-$ -side respectively. (As usual, the $+$ -side of an oriented curve lies to the left, and the $-$ -side lies to the right, if one traverses the curve according to its orientation.)

Let

$$I = \int_{-1}^1 (\sqrt{t^4 - 1})_+ dt + \int_\gamma (\sqrt{t^4 - 1})_+ dt.$$

Since $(\sqrt{t^4 - 1})_+ = -(\sqrt{t^4 - 1})_-$, for $t \in [-1, 1] \cup \gamma$, we have

$$2I = \int_{[-1, 1] \cup \gamma} \left[(\sqrt{t^4 - 1})_+ - (\sqrt{t^4 - 1})_- \right] dt = \oint_{\Gamma_{i,j} \cup \Gamma_{l,k}} \sqrt{z^4 - 1} dz,$$

where $\Gamma_{i,j}$ and $\Gamma_{l,k}$ are two closed contours encircling respectively the curve $[-1, 1]$ and γ once in the clockwise direction. After the contour deformation we pick up residue at $z = \infty$. We get

$$I = \frac{1}{2} \oint_{\Gamma_{i,j} \cup \Gamma_{l,k}} \sqrt{z^4 - 1} dz = \pm i\pi \text{res}_{\infty} \left(\sqrt{z^4 - 1} \right) = 0.$$

By the other hand, it is straightforward that $\Re \int_{-1}^1 \left(\sqrt{t^4 - 1} \right)_+ dt = 0$, which implies that

$$\Re \int_{\gamma} \left(\sqrt{t^4 - 1} \right)_+ dt = 0.$$

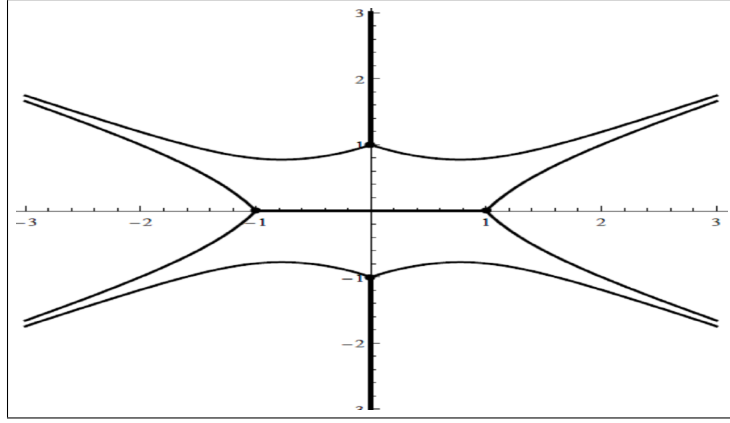


Figure 3: Critical graph of the quadratic differential $Q(z) = -(z^4 - 1) dz^2$.

The quadratic differential $\varphi(z) dz^2$ defines a φ -metric with the differential element $\sqrt{|\varphi(z)|} |dz|$. If γ is a rectifiable arc in \mathbb{C} , then its φ -length is defined by

$$|\gamma|_{\varphi} = \int_{\gamma} \sqrt{|\varphi(z)|} |dz|.$$

A trajectory of $\varphi(z) dz^2$ is finite if, and only if its φ -length is finite, otherwise is infinite. In particular, a critical trajectory is finite if and only if its two end points each are finite critical point.

Two Jordan arcs $\alpha, \beta : [0, 1] \rightarrow \mathbb{C}$ joining a point p_1 to a point p_2 in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$ are homotopic if there exists a continuous function $H : [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{\text{poles of } \varphi\}$ such that

$$\begin{cases} H(t, 0) = \alpha(t) \\ H(t, 1) = \beta(t) \end{cases}, t \in [0, 1].$$

It is an equivalence relation on the set \mathcal{J}_{p_1, p_2} of all Jordan arcs joining p_1 to p_2 in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$. If $\text{card}\{\text{poles of } \varphi\} = m \in \mathbb{N}$, then, it is well known that $\mathbb{C} \setminus \{\text{poles of } \varphi\}$ and the wedge of m circles have the same type of homotopy; in particular, there are 2^m classes of equivalence of the relation "homotopic" on \mathcal{J}_{p_1, p_2} .

Definition 4 *A locally rectifiable (in the spherical metric) curve γ_0 is called a φ -geodesic if it is locally shortest in the φ -metric. It is called critical geodesic if it is φ -geodesic and passing through a critical point of the quadratic differential $\varphi(z) dz^2$.*

Proposition 5 ([13, Theorem 16.2]) *Let γ be a φ -geodesic arc joining p_1 to p_2 in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$. Then for every $\gamma_1 \in \mathcal{J}_{p_1, p_2}$ which is homotopic to γ on $\mathbb{C} \setminus \{\text{poles of } \varphi\}$, we have $|\gamma_1|_\varphi \geq |\gamma|_\varphi$, with equality if and only if $\gamma_1 = \gamma$.*

We finish this section by the so-called Teichmüller Lemma that will be in use in the next section.

Definition 6 *A domain in \mathbb{C} bounded only by segments of φ -geodesic and/or horizontal and/or vertical trajectories of the quadratic differential $\varphi(z) dz^2$ (and their endpoints) is called φ -polygon.*

Lemma 7 (Teichmüller) *Let Ω be a φ -polygon, and let z_j be the singular points of $\varphi(z) dz^2$ on the boundary $\partial\Omega$ of Ω , with multiplicities n_j , and let $\theta_j \in [0, 2\pi]$ be the corresponding interior angles with vertices at z_j , respectively. Then*

$$\sum \left(1 - \theta_j \frac{n_j + 2}{2\pi}\right) = 2 + \sum n_i, \quad (6)$$

where n_i are the multiplicities of the singular points inside Ω .

3 Proofs

Lemma 8 *In the notation of Proposition 2*

- (a) *there exists at most one unbroken short trajectory of the quadratic differential $\varpi(P_t, z)$ connecting $a(t)$ and $b(t)$.*
- (b) *If there exist two short trajectories of the quadratic differential $\varpi(R_t, z)$ connecting $a(t)$ and $b(t)$, then they are not homotopic in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$.*

Proof.

- (a) Suppose that γ_1 and γ_2 are two unbroken short trajectories of $\varpi(P_t, z)$ connecting $a(t)$ and $b(t)$, and let Ω be the ϖ -polygon with vertices $a(t)$ and $b(t)$, and edges γ_1 and γ_2 . From Lemma 7, the left-hand side of (6) is smaller than 2, whereas the righthand side is clearly at least 2, a contradiction.
- (b) In the same vein of the previous proof, by taking γ_1 and γ_2 are two short trajectories of $\varpi(R_t, z)$ connecting $a(t)$ and $b(t)$; the fact that are homotopic in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$ means that there is no pole of R_t inside Ω ; and again, we get a contradiction with Lemma 7.

■

Remark 9 *The number of unbroken short geodesics of $\varpi_P(t, z)$ can be any integer between $\deg(P_t(z)) - 1$ and $\binom{2}{\deg(P_t(z))}$. We refer the reader to [11] for the proof.*

Remark 10 *It is well known that, by using 3 wedged circles, there are 8 homotopy classes in $\mathcal{J}_{a(t), b(t)}$. With the same way of the previous proof, and by Proposition 5, there exist at most 8 unbroken short geodesics of $\varpi(R_t, z)$ joining $a(t)$ and $b(t)$.*

Proof of Proposition 1. Let us denote Λ the subset of Ω formed by all t such that there exists a short trajectory of $\varpi(R_t, z)$ homotopic to ϕ_t in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$.

Let $t_0 \in \Lambda$. By continuity of the quadratic differential $\varpi(R_t, z)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t \in \Omega$ satisfying $|t - t_0| < \delta$, there exists a trajectory of $\varpi(R_t, z)$, say γ_t , emanating from $a(t)$ and intersecting the ε -neighborhood \mathcal{U}_ε of $b(t)$. If γ_t does not pass through $b(t)$, then, we may assume that $\delta > 0$ is small enough so that γ_t is intersected by an orthogonal trajectory σ_t emanating from $b(t)$ in some point $c(t)$. We denote by φ_t the path that follows the arc of γ_t from $a(t)$ to $c(t)$ and then continues to $b(t)$ along σ_t . Clearly the arcs ϕ_t and φ_t are homotopic in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$, and, by definition of orthogonal trajectories, the real part of the integral along φ_t of $\sqrt{R_t(z)}$ cannot vanish. This contradiction shows that the whole small neighborhood of t_0 still in Λ , and then Λ is an open subset of Ω .

Suppose now that (t_n) is sequence of Λ converging to $t \in \Omega$, so that $a(t_n)$ and $b(t_n)$ converge respectively to a and b . For each t_n , there exists

a unique short trajectory γ_n joining $a(t_n)$ and $b(t_n)$, and all the γ_n are homotopic to ϕ_{t_n} in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$. It is obvious that the limit set of the sequence γ_n (in the Hausdörff metrics) is either another short trajectory connecting a and b , or a union of two infinite critical trajectories γ_a and γ_b emanating respectively from a and b , and each of them diverges to some pole of the quadratic differential $\varpi(R_t, z)$. If γ_a and γ_b do not diverge to the same pole, or one of them diverges to a simple pole, then

$$\inf_{x \in \gamma_a, y \in \gamma_b} |x - y| = \text{dist}(\gamma_a, \gamma_b) > 0,$$

which contradicts the fact that $\lim_{n \rightarrow \infty} \gamma_n = \gamma_a \cup \gamma_b$. Let $c \in \{a_1, a_2, a_3\} \cup \{\infty\}$ be the common pole of divergence of γ_a and γ_b .

If c is a double pole (we assume without loss of generality that the residue of the quadratic differential $\varpi(R_t, z)$ at the pole c is non real, and then γ_a and γ_b diverge to c in log-spiral). Let σ be an orthogonal trajectory that diverges (of course, in log-spiral) to c . Then σ intersects γ_a and γ_b infinitely many times. Considering three consecutive points of intersection, it is obvious that we can construct two paths γ and γ' joining a and b formed by the three parts, from γ_a , σ and γ_b . Clearly, γ and γ' are not homotopic in $\mathbb{C} \setminus \{c\}$, and by continuity of the family ϕ_{t_n} , one of them must be homotopic to ϕ_{t_n} for $n \geq n_0$ for some integer n_0 . Then we get

$$\Re \int_{\gamma} \sqrt{R_t(z)} dz \neq 0, \text{ and } \Re \int_{\gamma'} \sqrt{R_t(z)} dz \neq 0,$$

which contradicts (3). Then, the limit set of the sequence γ_n is a short trajectory joining $a(t)$ and $b(t)$, and Λ is a closed subset of Ω . The cases when the residues at c are real (positive or negative) are in the same vein.

Finally, since Ω is a connected subset of \mathbb{C} , we conclude that either $\Lambda = \Omega$, or $\Lambda = \emptyset$. ■

Proof of Proposition 2. In order to discuss the possible existence of a short trajectory of $\varpi(P_t, z)$ connecting $a(t)$ and $b(t)$ for some $t \in \Omega$, we denote by $\Gamma_{a(t)}$ and $\Gamma_{b(t)}$ the sets of the three critical trajectories that emanate respectively from $a(t)$ and $b(t)$, and we consider the euclidian distance

$$\text{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) = \inf_{x \in \Gamma_{a(t)}, y \in \Gamma_{b(t)}} |x - y|.$$

Then we claim the following : The quadratic differential $\varpi(P_t, z)$ has a short trajectory connecting $a(t)$ and $b(t)$, if and only if, $\text{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) = 0$. Indeed, there are $n + 2$ asymptotic directions, spacing with equal angle $\frac{2\pi}{n+2}$ that can take any horizontal (resp. vertical) trajectory of the

quadratic differential $\varpi(P_t, z)$ diverging to infinity; the asymptotic directions of vertical trajectories are obtained by rotation of angle $\frac{\pi}{2}$. Obviously, if $\text{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) > 0$, then there is no short trajectory connecting $a(t)$ and $b(t)$. Assume that $\text{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) = 0$ and no short trajectory connects $a(t)$ and $b(t)$. Since $\Gamma_{a(t)} \cap \Gamma_{b(t)} = \emptyset$, there exist two horizontal trajectories $\gamma_{a(t)}$ and $\gamma_{b(t)}$ that emanate from $a(t)$ and $b(t)$ and diverge to infinity in the same direction D ; let σ be a vertical trajectory (not critical) diverging to infinity in the two directions adjacent to D . Obviously, σ intersects $\gamma_{a(t)}$ and $\gamma_{b(t)}$ in exactly two points $P_{a(t)}$ and $P_{b(t)}$. Let $\gamma \in \mathcal{J}_{a(t), b(t)}$ be the union of the part of $\gamma_{a(t)}$ from $a(t)$ to $P_{a(t)}$, and the part of σ from $P_{a(t)}$ to $P_{b(t)}$, and finally, the part of $\gamma_{b(t)}$ from $P_{b(t)}$ to $b(t)$. Integrating along γ , and since

$$\Re \int_{a(t)}^{P_{a(t)}} \sqrt{P_t(z)} dz = \Re \int_{P_{b(t)}}^{b(t)} \sqrt{P_t(z)} dz = 0,$$

we get

$$\Re \int_{\gamma} \sqrt{P_t(z)} dz = \Re \int_{P_{a(t)}}^{P_{b(t)}} \sqrt{P_t(z)} dz \neq 0,$$

which violates (4). By continuity of the function $t \mapsto \text{dist}(\Gamma_{a(t)}, \Gamma_{b(t)})$, it follows that the set of all $t \in \Omega$ such that the quadratic differential $\varpi(P_t, z)$ has no short trajectory connecting $a(t)$ and $b(t)$, is an open subset of Ω . Notice that Proposition 2 still valid with polynomials Q with higher degree or multiplicities of zeros. ■

4 Connection with Laguerre and Jacobi polynomials

The rescaled generalized Laguerre polynomials $L_n^{nC}(nz)$ with varying parameters nC , and the Jacobi polynomials $P_n^{(nA, nB)}(z)$ with varying parameters nA and nB can be given explicitly respectively by (see [14]):

$$L_n^{nC}(nz) = \sum_{k=0}^n \binom{n+nC}{n-k} \frac{(-z)^k}{k!},$$

$$P_n^{(nA, nB)}(z) = 2^{-n} \sum_{k=0}^n \binom{n+nA}{n-k} \binom{n+nB}{k} (z-1)^k (z+1)^{n-k}.$$

Jacobi or Laguerre polynomials with (real) parameters, depending on the degree n appear naturally as polynomial solutions of hypergeometric differential equations, or in the expressions of the wave functions of many classical systems in quantum mechanics; see [12].

With each polynomial p_n , we associate its normalized zero-counting measure μ_n ,

$$\mu_n = \mu(p_n) = \frac{\sum_{p_n(z)=0} \delta_z}{n}.$$

For a compact subset K in \mathbb{C} ,

$$\int_K d\mu_n = \frac{\text{number of zeros of } p_n \text{ in } K}{n}.$$

The zeros are counted with their multiplicities.

Following the works of Gonchar-Rakhmanov [3] and Stahl [4], it was shown that the sequence μ_n converges (as $n \rightarrow \infty$) in the weak-* topology to a measure, supported on short trajectories of related quadratic differentials. For the case of Laguerre, see [5],[1],[2]; for the case of Jacobi, see [6],[7],[9],[10].

The related quadratic differential for Laguerre polynomials is,

$$\varpi_C = -\frac{D_C(z)}{z^2} dz^2, \quad (7)$$

where

$$D_C(z) = z^2 - 2(C+2)z + C^2.$$

The zeros of $D_C(z)$ are

$$a(C) = C + 2 + 2\sqrt{C+1}, b(C) = C + 2 - 2\sqrt{C+1}. \quad (8)$$

The related quadratic differential for Jacobi polynomials is

$$\varpi_{A,B} = -\frac{D_{A,B}(z)}{(z^2-1)^2} dz^2, \quad (9)$$

where

$$D_{A,B}(z) = (A+B+2)^2 z^2 + 2(A^2 - B^2)z + (A-B)^2 - 4(A+B+1).$$

The zeros of $D_{A,B}(z)$ are

$$\begin{aligned} a(A,B) &= \frac{-A^2 + B^2 + 4\sqrt{(A+1)(B+1)(A+B+1)}}{(A+B+2)^2}, \\ b(A,B) &= \frac{-A^2 + B^2 - 4\sqrt{(A+1)(B+1)(A+B+1)}}{(A+B+2)^2}. \end{aligned}$$

Proposition 11 ([2]) Assume that $C \in \mathbb{C}_+$, and that γ is a Jordan arc connecting the zeros of $D_C(z)$ in the punctured plane $\mathbb{C} \setminus \{0\}$. Denote by $\sqrt{D_C(z)}$ the single-valued branch of this function in $\mathbb{C} \setminus \gamma$ determined by the condition

$$\sqrt{D_C(z)} \sim z, z \rightarrow \infty,$$

and let $\left(\sqrt{D_C(z)}\right)_+$ stand for its boundary values on the $+$ -side of γ . Then

$$\int_{\gamma} \frac{\left(\sqrt{D_C(t)}\right)_+}{t} dt \in \pm 2\pi i \{1, (C+1)\}. \quad (10)$$

Moreover, the integral in the left hand of (10) takes the value $\pm 2\pi i$ if and only if γ is such that it can be continuously deformed in $\mathbb{C} \setminus \{0\}$ to an arc not intersecting the positive real axis.

If we denote $\Omega = \{C \in \mathbb{C} : \Im C \geq 0\}$, and $R_C(z) = -\frac{D_C(z)}{z^2}$, then conditions (1),(2), and (3) are full-filled. Since it can be easily shown that for $C \in (-1, +\infty)$, the zeros $a(C)$ and $b(C)$ satisfy

$$0 < b(C) < a(C),$$

and the segment $[b(C), a(C)]$ is a short trajectory of the quadratic differential 7 (see Figure 4), we conclude the existence of the short trajectory for any $C \in \Omega$.

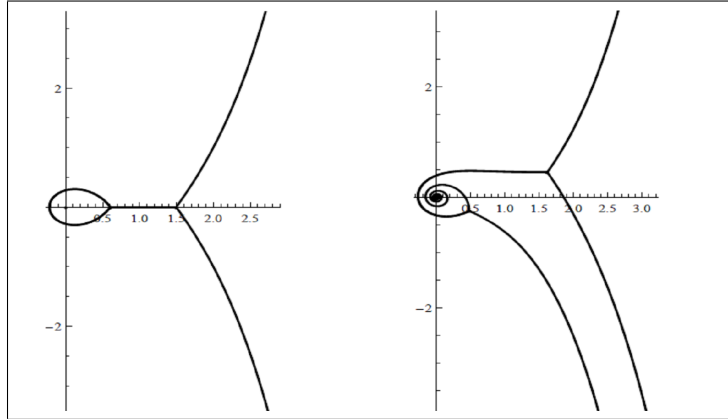


Figure 4: Critical graphs of $\varpi_{-0.95}$ (left) and $\varpi_{-0.95+0.1i}$ (right).

Proposition 12 ([6],[9]) *Let A, B satisfy assumptions*

$$A + 1 \neq 0, B + 1 \neq 0, A + B + 1 \neq 0, A + B + 2 \neq 0, \quad (11)$$

and let γ be a Jordan arc in $\mathbb{C} \setminus \{-1, 1\}$ joining the zeros of $D_{A,B}$, and $\sqrt{D_{A,B}}$ is its single-valued branch in $\mathbb{C} \setminus \gamma$ fixed by the condition

$$\sqrt{D_{A,B}}(z) \sim (A + B + 2)z, z \rightarrow \infty.$$

Then

$$\int_{\gamma} \frac{(\sqrt{D_{A,B}}(t))_{\pm}}{t^2 - 1} dt \in \pm 2\pi i \{1, (A + 1), (B + 1), (A + B + 1)\}, \quad (12)$$

where $(\sqrt{D_{A,B}}(t))_{\pm}$ is the boundary value on one of the sides of γ .

Moreover, if in addition of (11), $B > 0$, then the integral in the left hand side of (12) takes the value $\pm 2\pi i$ if and only if γ is such that conditions

$$\sqrt{D_{A,B}}(1) = 2A, \quad \sqrt{D_{A,B}}(-1) = -2B$$

are satisfied.

For $B > -1$ we denote

$$\Omega = \{A \in \mathbb{C} : A + 1 \neq 0, A + B + 1 \neq 0, A + B + 2 \neq 0\},$$

and $R_A(z) = -\frac{D_{A,B}(z)}{(z^2 - 1)^2}$, then conditions (1), (2), and (3) are satisfied. Taking into account that for $A \in \mathbb{R} \cap \Omega$, there exists a short trajectory of the quadratic differential 9, we conclude the existence of the short trajectory for any $A \in \Omega$. By repeating the same reasoning, we conclude the result for any A and B satisfying (11) (see Figures 5, 6).

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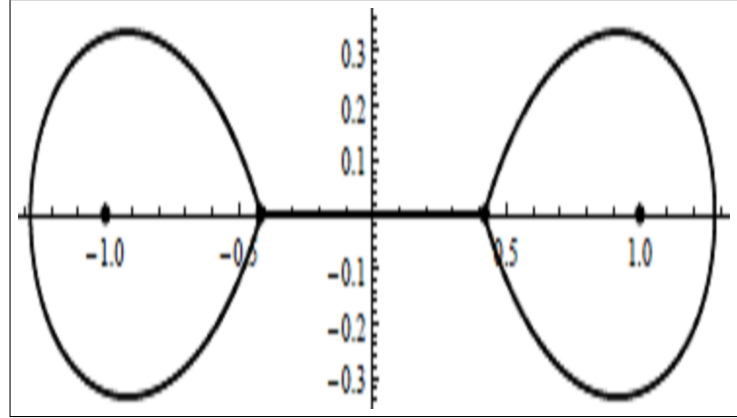


Figure 5: Critical graph of $\varpi_{10,10}$.

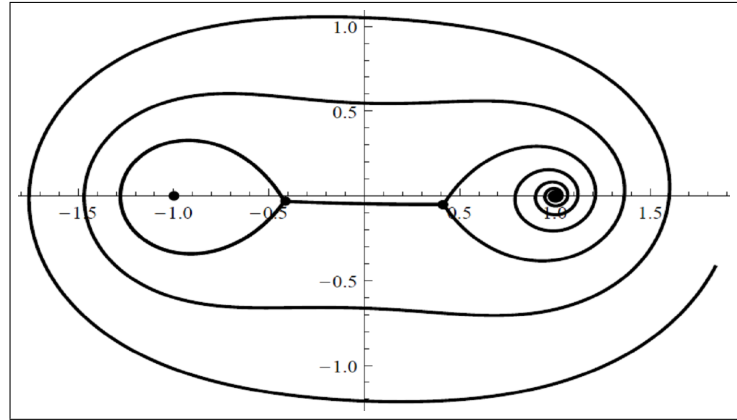


Figure 6: Critical graph of $\varpi_{10+i,10}$.

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